Higher

Mathematics

UNIT 3 OUTCOME 4

Wave Functions

Contents

HSN23400

OUTCOME 4

Wave Functions

1 Expressing *pcosx* + *qsinx* in the form *kcos*(*x* – *a*)

An expression of the form $p\cos x + q\sin x$ can be written in the form $k\cos(x-a)$ where

$$k = \sqrt{p^2 + q^2} \quad \text{and} \quad \tan a = \frac{k\sin a}{k\cos a}.$$

The following example shows how to achieve this.

EXAMPLES

1. Write $5\cos x° + 12\sin x°$ in the form $k\cos(x° - a°)$ where $0 \le a \le 360$.

Step 1

Expand $k\cos(x-a)$ using the compound angle formula.

$$\begin{aligned} &5\cos x° + 12\sin x° \\ &= k\cos(x° - a°) \\ &= k\cos x° \cos a° + k\sin x° \sin a° \end{aligned}$$

Step 2

Rearrange to compare with $p\cos x + q\sin x$.

$$= \underbrace{(k\cos a°)}_{5}\cos x° + \underbrace{(k\sin a°)}_{12}\sin x°$$

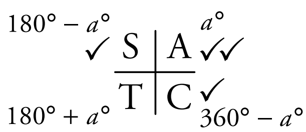
Step 3

Compare the coefficients of $\cos x$ and $\sin x$ with $p\cos x + q\sin x$.

$$k\cos a° = 5$$
$$k\sin a° = 12$$

Step 4

Mark the quadrants on a CAST diagram, according to the signs of $k\cos a$ and $k\sin a$.



Step 5

Find k and a using the formulae above (a lies in the quadrant marked twice in *Step 4*).

$$\begin{aligned} k &= \sqrt{5^2 + 12^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} \tan a° &= \frac{k\sin a°}{k\cos a°} \\ &= \tfrac{12}{5} \\ a &= \tan^{-1}\left(\tfrac{12}{5}\right) \\ &= 67{\cdot}4 \quad \text{(to 1 d.p.)} \end{aligned}$$

Step 6

State $p\cos x + q\sin x$ in the form $k\cos(x-a)$ using these values.

$$5\cos x° + 12\sin x° = 13\cos(x° - 67{\cdot}4°)$$

2. Write $5\cos x - 3\sin x$ in the form $k\cos(x-a)$ where $0 \leq a \leq 2\pi$.

$$\begin{aligned}5\cos x - 3\sin x &= k\cos(x-a)\\ &= k\cos x\cos a + k\sin x\sin a\\ &= (k\cos a)\cos x + (k\sin a)\sin x.\end{aligned}$$

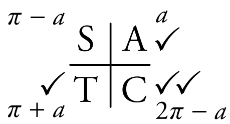
$k\cos a = 5$

$k\sin a = -3$

$k = \sqrt{5^2 + (-3)^2}$

$= \sqrt{34}$

$\tan a = \dfrac{k\sin a}{k\cos a} = -\frac{3}{5}$



Hence a is in the fourth quadrant.

First quadrant answer is:

$\tan^{-1}\left(\frac{3}{5}\right)$

$= 0{\cdot}540$ (to 3 d.p.).

So $a = 2\pi - 0{\cdot}540$

$= 5{\cdot}743$ (to 3 d.p.).

Note
Make sure your calculator is in radian mode.

Hence $5\cos x - 3\sin x = \sqrt{34}\cos(x - 5{\cdot}743)$.

2 Expressing *p*cos*x* + *q*sin*x* in other forms

An expression in the form $p\cos x + q\sin x$ can also be written in any of the following forms using a similar method:

$$k\cos(x+a), \qquad k\sin(x-a), \qquad k\sin(x+a).$$

EXAMPLES

1. Write $4\cos x^\circ + 3\sin x^\circ$ in the form $k\sin(x^\circ + a^\circ)$ where $0 \leq a \leq 360$.

$$\begin{aligned}4\cos x^\circ + 3\sin x^\circ &= k\sin(x^\circ + a^\circ)\\ &= k\sin x^\circ\cos a^\circ + k\cos x^\circ\sin a^\circ\\ &= (k\cos a^\circ)\sin x^\circ + (k\sin a^\circ)\cos x^\circ\end{aligned}$$

$k\cos a^\circ = 3$

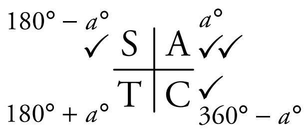
$k\sin a^\circ = 4$

$k = \sqrt{4^2 + 3^2}$

$= \sqrt{25}$

$= 5$

$\tan a^\circ = \dfrac{k\sin a^\circ}{k\cos a^\circ} = \frac{4}{3}$



Hence a is in the first quadrant.

So:

$a = \tan^{-1}\left(\frac{4}{3}\right)$

$= 53{\cdot}1$ (to 1 d.p.)

Hence $4\cos x^\circ + 3\sin x^\circ = 5\sin(x^\circ + 53{\cdot}1^\circ)$.

2. Write $\cos x - \sqrt{3}\sin x$ in the form $k\cos(x+a)$ where $0 \le a \le 2\pi$.

$$\begin{aligned}\cos x - \sqrt{3}\sin x &= k\cos(x+a)\\ &= k\cos x\cos a - k\sin x\sin a\\ &= (k\cos a)\cos x - (k\sin a)\sin x\end{aligned}$$

$k\cos a = 1$

$k\sin a = \sqrt{3}$

Hence a is in the first quadrant.

$$\begin{aligned}k &= \sqrt{1^2 + \left(-\sqrt{3}\right)^2}\\ &= \sqrt{1+3}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

$\tan a = \dfrac{k\sin a}{k\cos a} = \sqrt{3}$

So:

$$\begin{aligned}a &= \tan^{-1}\left(\sqrt{3}\right)\\ &= \tfrac{\pi}{3}\end{aligned}$$

Hence $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \tfrac{\pi}{3}\right)$.

3 Multiple Angles

We can use the same method with expressions involving the same multiple angle, i.e. $p\cos(nx) + q\sin(nx)$, where n is a constant.

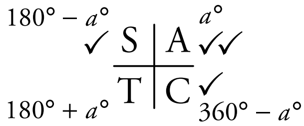
EXAMPLE

Write $5\cos 2x° + 12\sin 2x°$ in the form $k\sin(2x° + a°)$ where $0 \le a \le 360$.

$$\begin{aligned}5\cos 2x° + 12\sin 2x° &= k\sin(2x° + a°)\\ &= k\sin 2x°\cos a° + k\cos 2x°\sin a°\\ &= (k\cos a°)\sin 2x° + (k\sin a°)\cos 2x°\end{aligned}$$

$k\cos a° = 12$

$k\sin a° = 5$



Hence a is in the first quadrant.

$$\begin{aligned}k &= \sqrt{12^2 + 5^2}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

$\tan a° = \dfrac{k\sin a°}{k\cos a°} = \dfrac{5}{12}$

So:

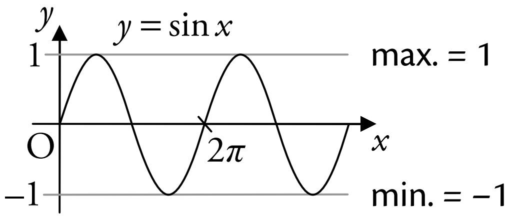
$$\begin{aligned}a &= \tan^{-1}\left(\tfrac{5}{12}\right)\\ &= 22{\cdot}6 \quad \text{(to 1 d.p.)}\end{aligned}$$

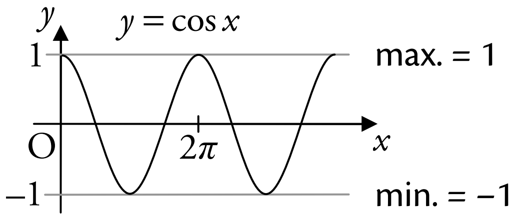
Hence $5\cos 2x° + 12\sin 2x° = 13\sin(2x° + 22{\cdot}6°)$.

4 Maximum and Minimum Values

To work out the maximum or minimum values of $p\cos x + q\sin x$, we can rewrite it as a single trigonometric function, e.g. $k\cos(x-a)$.

Recall that the maximum value of the sine and cosine functions is 1, and their minimum is -1.





EXAMPLE

Write $4\sin x + \cos x$ in the form $k\cos(x-a)$ where $0 \leq a \leq 2\pi$ and state:

(i) the maximum value and the value of $0 \leq x < 2\pi$ at which it occurs

(ii) the minimum value and the value of $0 \leq x < 2\pi$ at which it occurs.

$$\begin{aligned} 4\sin x + \cos x &= k\cos(x-a) \\ &= k\cos x\cos a + k\sin x\sin a \\ &= (k\cos a)\cos x + (k\sin a)\sin x \end{aligned}$$

$k\cos a = 1$

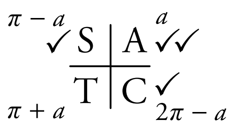
$k\sin a = 4$

$$\begin{aligned} k &= \sqrt{1^2+4^2} \\ &= \sqrt{17} \end{aligned}$$

$$\tan a = \frac{k\sin a}{k\cos a} = 4$$

So:

$$\begin{aligned} a &= \tan^{-1}(4) \\ &= 1{\cdot}326 \quad \text{(to 3 d.p.)} \end{aligned}$$



Hence a is in the first quadrant.

Hence $4\sin x + \cos x = \sqrt{17}\cos(x - 1{\cdot}326)$.

The maximum value of $\sqrt{17}$ occurs when:

$$\begin{aligned} \cos(x-1{\cdot}326) &= 1 \\ x - 1{\cdot}326 &= \cos^{-1}(1) \\ x - 1{\cdot}326 &= 0 \\ x &= 1{\cdot}326 \quad \text{(to 3 d.p.).} \end{aligned}$$

The minimum value of $-\sqrt{17}$ occurs when:

$$\begin{aligned} \cos(x-1{\cdot}326) &= -1 \\ x - 1{\cdot}326 &= \cos^{-1}(-1) \\ x - 1{\cdot}326 &= \pi \\ x &= 4{\cdot}468 \quad \text{(to 3 d.p.).} \end{aligned}$$

5 Solving Equations

The method of writing two trigonometric terms as one can be used to help solve equations involving both a $\sin(nx)$ and a $\cos(nx)$ term.

EXAMPLES

1. Solve $5\cos x° + \sin x° = 2$ where $0 \le x \le 360$.

First, we write $5\cos x° + \sin x°$ in the form $k\cos(x° - a°)$:

$$\begin{aligned} 5\cos x° + \sin x° &= k\cos(x° - a°) \\ &= k\cos x° \cos a° + k\sin x° \sin a° \\ &= (k\cos a°)\cos x° + (k\sin a°)\sin x° \end{aligned}$$

$k\cos a° = 5$

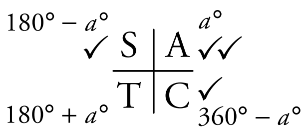
$k\sin a° = 1$

$$\begin{aligned} k &= \sqrt{5^2 + 1^2} \\ &= \sqrt{26} \end{aligned}$$

$$\tan a° = \frac{k\sin a°}{k\cos a°} = \tfrac{1}{5}$$

So:

$$\begin{aligned} a &= \tan^{-1}\left(\tfrac{1}{5}\right) \\ &= 11{\cdot}3 \quad \text{(to 1 d.p.)} \end{aligned}$$

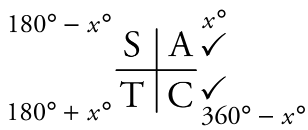


Hence a is in the first quadrant.

Hence $5\cos x° + \sin x° = \sqrt{26}\cos(x° - 11{\cdot}3°)$.

Now we use this to help solve the equation:

$$\begin{aligned} 5\cos x° + \sin x° &= 2 \\ \sqrt{26}\cos(x° - 11{\cdot}3°) &= 2 \\ \cos(x° - 11{\cdot}3°) &= \tfrac{2}{\sqrt{26}} \end{aligned}$$



$$\begin{aligned} x - 11{\cdot}3 &= \cos^{-1}\left(\tfrac{2}{\sqrt{26}}\right) \\ &= 66{\cdot}9 \quad \text{(to 2 d.p.)} \end{aligned}$$

$$\begin{aligned} x - 11{\cdot}3 &= 66{\cdot}9 \quad \text{or} \quad 360 - 66{\cdot}9 \\ x - 11{\cdot}3 &= 66{\cdot}9 \quad \text{or} \quad 293{\cdot}1 \\ x &= 78{\cdot}2 \quad \text{or} \quad 304{\cdot}4. \end{aligned}$$

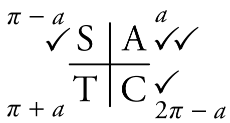
2. Solve $2\cos 2x + 3\sin 2x = 1$ where $0 \le x \le 2\pi$.

First, we write $2\cos 2x + 3\sin 2x$ in the form $k\cos(2x - a)$:

$$\begin{aligned} 2\cos 2x + 3\sin 2x &= k\cos(2x - a) \\ &= k\cos 2x\cos a + k\sin 2x\sin a \\ &= (k\cos a)\cos 2x + (k\sin a)\sin 2x \end{aligned}$$

$k\cos a = 2$

$k\sin a = 3$



Hence a is in the first quadrant.

$$\begin{aligned} k &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13} \end{aligned}$$

$\tan a = \dfrac{k\sin a}{k\cos a} = \frac{3}{2}$

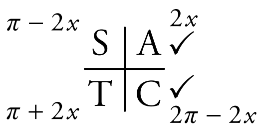
So:

$$\begin{aligned} a &= \tan^{-1}\left(\tfrac{3}{2}\right) \\ &= 0{\cdot}983 \quad \text{(to 3 d.p.)} \end{aligned}$$

Hence $2\cos 2x + 3\sin 2x = \sqrt{13}\cos(2x - 0{\cdot}983)$.

Now we use this to help solve the equation:

$$\begin{aligned} 2\cos 2x + 3\sin 2x &= 1 \\ \sqrt{13}\cos(2x - 0{\cdot}983) &= 1 \\ \cos(2x - 0{\cdot}983) &= \tfrac{1}{\sqrt{13}} \end{aligned}$$



$0 < x < 2\pi$

$0 < 2x < 4\pi$

$$\begin{aligned} 2x - 0{\cdot}983 &= \cos^{-1}\left(\tfrac{1}{\sqrt{13}}\right) \\ &= 1{\cdot}290 \quad \text{(to 3 d.p.)} \end{aligned}$$

$2x - 0{\cdot}983 = 1{\cdot}290$ or $2\pi - 1{\cdot}290$

or $2\pi + 1{\cdot}290$ or $2\pi + 2\pi - 1{\cdot}290$

or ~~$2\pi + 2\pi + 1{\cdot}290$~~

$2x - 0{\cdot}983 = 1{\cdot}290$ or $4{\cdot}993$ or $7{\cdot}573$ or $11{\cdot}276$

$2x = 2{\cdot}273$ or $5{\cdot}976$ or $8{\cdot}556$ or $12{\cdot}259$

$x = 1{\cdot}137$ or $2{\cdot}988$ or $4{\cdot}278$ or $6{\cdot}130$

6 Sketching Graphs of *y = pcosx + qsinx*

Expressing $p\cos x + q\sin x$ in the form $k\cos(x-a)$ enables us to sketch the graph of $y = p\cos x + q\sin x$.

EXAMPLES

1. (a) Write $7\cos x° + 6\sin x°$ in the form $k\cos(x° - a°)$, $0 \le a \le 360$.

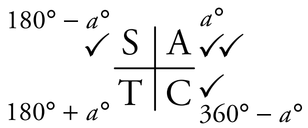
   (b) Hence sketch the graph of $y = 7\cos x° + 6\sin x°$ for $0 \le x \le 360$.

(a) First, we write $7\cos x° + 6\sin x°$ in the form $k\cos(x° - a°)$:

$$\begin{aligned}7\cos x° + 6\sin x° &= k\cos(x° - a°)\\ &= k\cos x°\cos a° + k\sin x°\sin a°\\ &= (k\cos a°)\cos x° + (k\sin a°)\sin x°\end{aligned}$$

$k\cos a° = 7$

$k\sin a° = 6$



Hence a is in the first quadrant.

$$\begin{aligned}k &= \sqrt{6^2 + 7^2}\\ &= \sqrt{36+49}\\ &= \sqrt{85}\end{aligned}$$

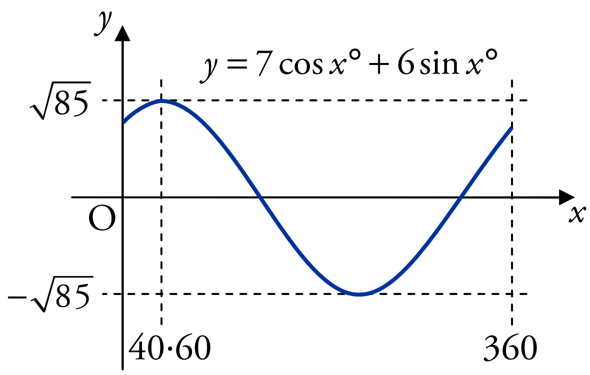
$$\tan a° = \frac{k\sin a°}{k\cos a°} = \frac{6}{7}$$

So:

$$\begin{aligned}a &= \tan^{-1}\left(\tfrac{6}{7}\right)\\ &= 40{\cdot}6 \quad \text{(to 1 d.p.)}\end{aligned}$$

Hence $7\cos x° + 6\sin x° = \sqrt{85}\cos(x° - 40{\cdot}6°)$.

(b) Now we can sketch the graph of $y = 7\cos x° + 6\sin x°$:



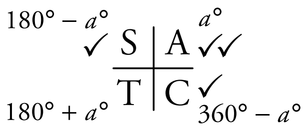
2. Sketch the graph of $y = \sin x° + \sqrt{3}\cos x°$ for $0 \le x \le 360$.

First, we write $\sin x° + \sqrt{3}\cos x°$ in the form $k\cos(x° - a°)$:

$$\begin{aligned}\sin x° + \sqrt{3}\cos x° &= k\cos(x° - a°)\\ &= k\cos x°\cos a° + k\sin x°\sin a°\\ &= (k\cos a°)\cos x° + (k\sin a°)\sin x°\end{aligned}$$

$k\cos a° = \sqrt{3}$

$k\sin a° = 1$



Hence a is in the first quadrant.

$$\begin{aligned}k &= \sqrt{1^2 + \sqrt{3}^2}\\ &= \sqrt{1+3}\\ &= 2\end{aligned}$$

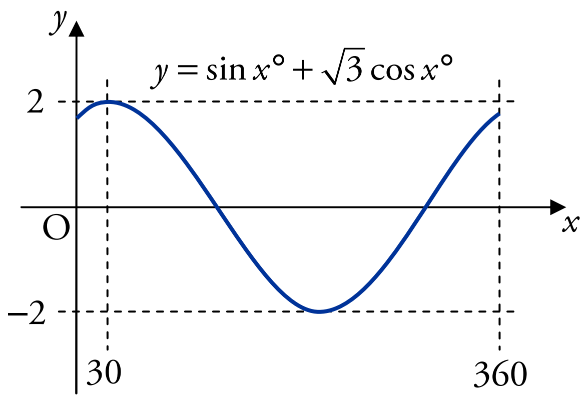
$$\tan a° = \frac{k\sin a°}{k\cos a°} = \frac{1}{\sqrt{3}}$$

So:

$$\begin{aligned}a &= \tan^{-1}\left(\tfrac{1}{\sqrt{3}}\right)\\ &= 30\end{aligned}$$

Hence $\sin x° + \sqrt{3}\cos x° = 2\cos(x° - 30°)$.

Now we can sketch the graph of $y = \sin x° + \sqrt{3}\cos x°$:



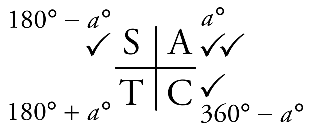
3. (a) Write $5\sin x° - \sqrt{11}\cos x°$ in the form $k\sin(x° - a°)$, $0 \le a \le 360$.

(b) Hence sketch the graph of $y = 5\sin x° - \sqrt{11}\cos x° + 2$, $0 \le x \le 360$.

(a) $$\begin{aligned} 5\sin x° - \sqrt{11}\cos x° &= k\sin(x° - a°) \\ &= k\sin x°\cos a° + k\cos x°\sin a° \\ &= (k\cos a°)\sin x° + (k\sin a°)\cos x° \end{aligned}$$

$k\cos a° = 5$

$k\sin a° = \sqrt{11}$



Hence a is in the first quadrant.

$$\begin{aligned} k &= \sqrt{5^2 + \sqrt{11}^2} \\ &= \sqrt{25 + 11} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

$$\tan a° = \frac{k\sin a°}{k\cos a°} = \frac{\sqrt{11}}{5}$$

So:

$$\begin{aligned} a &= \tan^{-1}\left(\frac{\sqrt{11}}{5}\right) \\ &= 33{\cdot}6 \quad \text{(to 1 d.p.)} \end{aligned}$$

Hence $5\sin x° - \sqrt{11}\cos x° = 6\sin(x° - 33{\cdot}6°)$.

(b) Now sketch the graph of

$y = 5\sin x° - \sqrt{11}\cos x° + 2 = 6\sin(x° - 33{\cdot}6°) + 2$: